The Chang Ideal

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Ideal Fun

"Ideals combine the 2 best things in set theory–forcing and elementary embeddings." —(PhD student in Münster)

Outline of Talk

1. Nonstationary ideal and generic ultrapowers
2. Chang's Conjecture
3. Chang Ideal
4. Consistency strength
   - 2 very different kinds of results
     4.1 Core model theory
     4.2 Foreman's results

NS_{ω_1}

The club filter on ω_1 is the collection of $Z \subset \omega_1$ such that Z contains a club.

- countably closed
- normal

The nonstationary ideal on ω_1 (denoted NS_{ω_1})is the dual of the club filter.

Generic ultrapower

Let $I := NS_{\omega_1}$.

- Define a partial order on $P(\omega_1)$ by: $A \subseteq_I B$ iff $A - B \in I$.
- Consider the poset $(\{\text{stationary subsets of } \omega_1\}, \subseteq_I)$.
- If $G \subset (\{\text{stationary subsets of } \omega_1\}, \subseteq_I)$ is generic, then it is an ultrafilter over V. (i.e. G is ultrafilter on $P^V(\omega_1)$)
  - So (from point of view of $V[G]$) there is the ultrapower map $V \to_G ult(V, G)$ and Los Theorem.

Generic ultrapower, cont.

Genericity of G implies that it inherits nice properties of I:

- G is countably complete w.r.t. V; i.e. if $\langle z_n | n \in \omega \rangle$ is element of V and every $z_n \in G$, then $\bigcap_{n \in \omega} z_n \in G$.
- G is normal w.r.t. V.

CAUTION: do not let "countably complete" mislead you; the poset is definitely NOT a countably complete poset.

Generic ultrapower, cont.

That forcing is equivalent to forcing with a certain boolean algebra $(P(\omega_1)/I - \{[\emptyset]_I\}, \leq_I)$ whose elements are equivalence classes.

- Sums in the boolean algebra correspond to diagonal unions
- Ideal is called saturated iff this boolean algebra is complete

Generic ultrapower, cont.

Interesting facts:

- $ult(V, G)$ *always* has a wellfounded initial segment which is isomorphic to ω_2; this is due to *normality* of I.
- $cr(j) = \omega_1^V$
- I is called precipitous iff for every generic G, $ult(V, G)$ is wellfounded. (note this is really a statement within V about the poset.)

General NS ideal (Shelah)

Fix a set **S** and let $A = \bigcup \mathbf{S}$. (typical situtation: $A = H_\theta$, **S** is some collection of $X \in H_\theta$ such that $X \prec H_\theta$)

- The strong club filter (on **S**) is the filter generated by collections of the form $C_{\mathcal{A}} := \{X \in \mathbf{S} | X \prec \mathcal{A}\}$ where $\mathcal{A}$ is some structure in a countable language on A.

- A set $T \subset \mathbf{S}$ is called (weakly) stationary iff it intersects every set in the strongly club filter
  - i.e. for every structure $\mathcal{A} = (H_\theta, \in, \ldots)$ there is an $X \in T$ such that $X \prec \mathcal{A}$.

General NS ideal, cont.

- EXAMPLE:
  - $\mathbf{S} := [H_\theta]^{\omega_1}$
  - $T := \{X \in \mathbf{S} | X \cap \omega_2 \in \omega_2 \cap cof(\omega)\}$
- EXAMPLE???:
  - $\mathbf{S} := [H_\theta]^{\omega_1}$
  - $T := \{X \in \mathbf{S} | |X \cap \omega_1| = \omega\}$. Is T (weakly) stationary?

  We'll return to this last example later

General NS ideal, cont.

The collection of nonstationary subsets of $\mathbf{S}$ is denoted $NS \restriction \mathbf{S}$.

For simplicity: only will consider $\mathbf{S}$ such that $\bigcup \mathbf{S} = H_\theta$ (e.g. $S = [H_\theta]^\omega$).

If $\mathbf{S}$ is itself weakly stationary then $NS \restriction \mathbf{S}$ is:

- countably complete (sometimes more)
- normal
  - i.e. for every regressive $F : \mathbf{S} \to V$ there is a weakly stationary set on which F is constant.

General NS ideal: generic ultrapower

Let $I := NS \restriction \mathbf{S}$ and force with $P(\mathbf{S})/I$.

- yields rich generic ultrapowers if the underlying set is rich (e.g. if $\bigcup \mathbf{S} = H_\theta$).
- Let $j : V \to_G ult(V, G)$
- $ult(V, G)$ is always wellfounded past θ!

Generic ultrapower, cont.

- $j \restriction H_\theta^V$ is always an element of $ult(V, G)$!
  - This is due to normality of I; you can show that $([id]_G, \in_G)$ is isomorphic to $(H_\theta^V, \in)$ via the transitive collapse of $[id]_G$ as seen by $ult(V, G)$.
  - Each $\nu \leq \theta$ in the generic ultrapower is represented by $X \mapsto otp(X \cap \nu)$.

Generic ultrapower, cont.

However, typically the *image* of the critical point of j lands in an illfounded part.

Chang's Conjecture

Definition

Chang's Conjecture, written $(\omega_2, \omega_1) \twoheadrightarrow (\omega_1, \omega)$ is the statement that for every structure $\mathcal{A} = (\omega_2, (f_n)_{n\in\omega})$ there is an $X \prec \mathcal{A}$ with $|X| = \omega_1$ and $|X \cap \omega_1| = \omega$.

- Generalization of Löwenheim-Skolem Theorem
- equivalent to requiring the structures to be on H_θ (some $\theta \geq \omega_2$).
- Obvious generalizations to other cardinals

The Chang Ideal

Assume Chang's Conjecture holds. Fix large θ and let
$\mathbf{S} := \{X \prec H_\theta | X \text{ is a Chang structure}\}$.

The Chang Ideal is $NS \restriction \mathbf{S}$.

Generic ultrapower by a Chang Ideal

Let I be the Chang ideal (at some large H_θ) and G generic for the corresponding p.o.

The image of the critical point is *always* in the wellfounded part of a Chang generic ultrapower.

- in fact $j(\omega_1^V)$ is always ω_2^V.

Consistency Strength of Chang's Conjecture

$(\omega_2, \omega_1) \twoheadrightarrow (\omega_1, \omega)$ equiconsistent with ω_1-Erdös cardinal (Silver; Donder)

Consistency Strength of Chang's Conjecture, cont.

[show how to get 0-sharp] [Condensation Lemma for L is key]

What about $(\omega_3, \omega_2) \twoheadrightarrow (\omega_2, \omega_1)$?

UPPER BOUNDS: Consistent relative to huge cardinals (Laver; Kunen)

LOWER BOUNDS:

- (C.) Implies there is inner model with repeat measures (builds on earlier work of Koepke, Vickers,...)
- (Schindler) Assuming CH, model of $o(\kappa) = \kappa^{+\omega}$.

.

Aside: saturated ideals

We say NS_{ω_1} is saturated iff all antichains in $P(\omega_1)/NS$ have size $< \omega_2$.

- equiconsistent with Woodin cardinal (Steel; Shelah)

Precipitousness of Chang ideal

Recently, Schindler showed that the consistency power of a saturated ideal comes merely from its precipitousness and the fact that $\Vdash j_{\dot{G}}(\omega_1^V) = \omega_2^V$.

- $\Vdash_{Changideal} j_{\dot{G}}(\omega_1^V) = \omega_2^V$
- So if Chang ideal is *precipitous*, then by Schindler's result there is inner model with Woodin.
- This is optimal, b/c if there is a Woodin cardinal then there is a forcing which makes Chang Ideal precipitous (F-M-S)

Results of Foreman

Chang Ideal Condensation (CIC): "Chang's Conjecture holds and there are many structures for which the Chang ideal condenses nicely"

Theorem

(Foreman). CON(ZFC + 2-huge) $\implies$ CON(CIC) $\implies$ CON(ZFC + 1-huge).

Results of Foreman, cont.

Foreman's arguments involve his notion of a *decisive* ideal. (decisiveness is defined in terms of generic elementary embeddings).

Main research goal

How are ideals related to large cardinals in inner models?

This question has a long history with good results; but far from solved.

Possibly more detail?

- covering arguments for $(\omega_3, \omega_2) \twoheadrightarrow (\omega_2, \omega_1)$.